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Application of Bernstein collocation method for solving the generalized regularized long wave equations

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The regularized and the modified regularized long wave (RLW and MRLW) equations are solved numerically by the Bernstein polynomials in both the space and time directions based on Kronecker product. In this paper, we applied a fully different Bernstein collocation method than the other methods which used Bernstein polynomials to solve the problems. The approximate solution is defined by the Bernstein polynomials in all directions. A general form for any m derivative of any Bernstein polynomials is constructed. A general matrix form for the vector of any m derivative of any Bernstein polynomials is also constructed. Convergence study for the proposed numerical scheme is investigated. To determine the conservation properties of the RLW and MRLW equations, three invariants of motion (I_1, I_2 and I_3) are computed. To test the accuracy, two error norms ($\|E\|_2$ and $\|E\|_\infty$) are evaluated. Numerical outcomes and comparisons with other techniques for the single and the interaction of two solitary waves for RLW and MRLW equations are presented.

1. Introduction

Nonlinear partial differential equations (NPDEs) have been one of the essential tools for modeling most real phenomena in science and engineering such as the regularized long wave (RLW) equation and the modified regularized long wave (MRLW) equation. Peregrine [1] was the first who presented the RLW and MRLW equation as a model for small amplitude long waves on the surface of water in a channel. The RLW and MRLW equation are used to describe waves in plasma, shallow water and phonon packets in nonlinear crystals… etc.

In this paper, we consider the following form of the generalized regularized long wave (GRLW) equation [4,11]:

$$\frac{\partial u}{\partial t} + [1 + q(q+1)u^q]\frac{\partial u}{\partial x} - \mu\frac{\partial^3 u}{\partial x^2 \partial t} = 0,$$
$$(x,t) \in [x_0, x_{M_x}] \times [0, t_{M_t}]. \tag{1.1}$$

The analytical value of Eq. (1.1) is given by [4,11]:

$$u(x,t) = A\left[\mathrm{sech}^2[BW(x - x_c - (\upsilon+1)t)]\right]^{\frac{1}{q}}, \tag{1.2}$$

with amplitude $A = \left[\frac{\upsilon(q+2)}{2q}\right]^{\frac{1}{q}}$ and the inverse width $BW = \frac{q}{2}Q$, where $Q = \sqrt{\frac{\upsilon}{\mu(\upsilon+1)}}$.

Eq. (1.1) is called the RLW equation and the MRLW equation if $q = 1$ and $q = 2$, respectively. Some of the previous studies on the RLW and MRLW equations are in [2–8,11–28], and references therein. Zeybek and Karakoc [2] presented lumped Galerkin approach with cubic B-spline to solve the GRLW equation. Zheng et al. [3] studied the barycentric interpolation collocation mehod to solve the GRLW equation. Hammad and El-Azab [4] used Chebyshev–Chebyshev spectral collocation method (C-C SCM) to solve GRLW equation. Akbari and Mokhtari [5] presented a compact finite difference method to solve the generalized long wave equation. Guo et al. [6] used the element-free kp-Ritz method to solve the GRLW equation. Huang and Zhang [7] obtained the element-free approximation of GRLW equation. Mohammadi [8] obtained the numerical solution of the GRLW equation by using the exponential B-spline collocation method. Hammad and El-Azab [11] used a 2 N order compact finite difference method (CFDM) to solve

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GRLW equation. Karakoc and Zeybek [12] used septic B-spline collocation method to solve GRLW equation. Hassan [13] used Fourier spectral method to solve MRLW equation. There are a lot of another works for the GRLW equations, such as: Petrov–Galerkin finite element method [16], Riccati – Bernoulli sub-ODE method and subdomain finite element method [19], finite difference approach for time derivatives and deltashaped basis functions for space discretization [21], a collocation algorithm based on quintic B-splines [22], four local momentum-preserving algorithms [23], travelling wave solutions [25], semi-analytical method with a new fractional derivative operator [26], the impact of LRBF FD on the solutions [27] and conservative difference scheme of solitary wave solutions [28]. Also, there are a lot of another works for the MRLW equation, such as: collocation of quintic B-splines over the finite elements [14], a septic B-spline collocation method [15], cubic B-spline Galerkin finite element method [17], Petrov-Galerkin finite element method [18] and B spline environment [20]. Jhangeer et al. [24] used analytical and numerical approaches for the perturbed and unperturbed fractional RLW equation.

Our main purpose in this paper is to improve and apply the Bernstein collocation method (B-CM) based on Kronecker product for solving the RLW and MRLW equations. Firstly, we discretize the space and time directions by Bernstein polynomials, then we get system of nonlinear equations which are solved numerically by Newton – Raphson method. There are many researchers used the Bernstein polynomials collocation method (BPCM) for the solution of the NPDEs, integral and integro-differential equations as in [9,10] and references therein, also, there are another works based on Bernstein polynomials such as [29–31].

The paper is coordinated in six sections as follows: In Section 2, a Bernstein collocation method (B-CM) is presented and the proposed scheme is constucted by using Kronecker and Hadamard products, also through this section, we constructed a general form for any m derivative of any Bernstein polynomials and we also constructed a general matrix form for the vector of any m derivative of any Bernstein polynomials. In Section 3, a B-CM is applied to solve the RLW and MRLW equations. In Section 4, convergence study for the proposed numerical scheme is investigated. Numerical results for solving the RLW and MRLW equation, the single solitary wave and the interaction of two solitary waves for our both problems are presented, also numerical comparisons with other methods are presented in Section 5. Finally, a conclusion is given at the end of the paper in Section 6.

2. Numerical methodology

Through this section, the Bernstein polynomials and their derivatives are determined. The general form for any derivative of any Bernstein polynomials is introduced for the first time in this paper, also a general matrix form of a vector of any derivative of any Bernstein polynomials is constructed. The implementation of Bernstein collocation method is done for getting the approximate solution of the RLW and MRLW by using Kronecker and Hadamard product which are used to represent all the equations in a matrix form.

Firstly, consider mesh points (x_i, t_j) in the region $[x_0, x_{M_x}] \times [0, t_{M_t}]$ are defined by

$$x_i = x_0 + ih_x, \qquad h_x = x_{i+1} - x_i = \frac{x_{M_x} - x_0}{M_x}, \qquad 0 \le i \le M_x.$$

$$t_j = jh_t, \quad h_t = t_{j+1} - t_j = \frac{t_{M_t}}{M_t}, \quad 0 \le j \le M_t.$$

2.1. Bernstein polynomials

The general form of the Bernstein polynomials of degree M_x on the interval $[x_0, x_{M_x}]$ are defined [9,10] by

$$B_{i,M_x}(x) = \binom{M_x}{i} \frac{(x - x_0)^i (x_{M_x} - x)^{M_x - i}}{(x_{M_x} - x_0)^{M_x}}, \quad i = 0, 1, \cdots, M_x, \tag{2.1}$$

where the binomial coefficients are given by

$$\binom{M_x}{i} = \frac{M_x!}{i!(M_x - i)!}.$$

There are $M_x + 1$ polynomials with degree M_x satisfy the following properties:

$$B_{i,M_x}(x) = 0, \quad \text{if} \quad i < 0 \quad \text{or} \quad i > M_x,$$

$$B_{i,M_x}(x_0) = B_{i,M_x}(x_{M_x}) = 0, \quad \text{for} \quad 1 \le i \le M_x - 1, \tag{2.2}$$

$$\sum_{i=0}^{M_x} B_{i,M_x}(x) = 1.$$

The Bernstein polynomials form a complete basis over the interval $[x_0, x_{M_x}]$, we can show that any given polynomial of degree M_x can be expressed in terms of linear combination of the basis functions. The recurrence relation and the derivatives of the Bernstein polynomials are given by

$$B_{i,M_x}(x) = \frac{x_{M_x} - x}{x_{M_x} - x_0} B_{i,M_x-1}(x) + \frac{x - x_0}{x_{M_x} - x_0} B_{i-1,M_x-1}(x), \tag{2.3}$$

$$B'_{i,M_x}(x) = \frac{M_x}{x_{M_x} - x_0} \left[B_{i-1,M_x-1}(x) - B_{i,M_x-1}(x)\right], \tag{2.4}$$

$$B''_{i,M_x}(x) = \frac{M_x(M_x - 1)}{(x_{M_x} - x_0)^2} \left[B_{i-2,M_x-2}(x) - 2B_{i-1,M_x-2}(x) + B_{i,M_x-2}(x)\right], \tag{2.5}$$

$$B'''_{i,M_x}(x) = \frac{M_x(M_x - 1)(M_x - 2)}{(x_{M_x} - x_0)^3} \left[B_{i-3,M_x-3}(x) - 3B_{i-2,M_x-3}(x) + 3B_{i-1,M_x-3}(x) - B_{i,M_x-3}(x)\right], \tag{2.6}$$

and hence, the m derivative of any Bernstein polynomials is given by

$$B^{(m)}_{i,M_x}(x) = \frac{\prod_{k=1}^{m}(M_x - k + 1)}{(x_{M_x} - x_0)^m} \left[\sum_{j=0}^{m} \binom{m}{j} (-1)^j B_{i-m+j,M_x-m}(x)\right], \tag{2.7}$$

we can rewrite any m derivative of any Bernstein polynomials in a matrix form as

$$\begin{bmatrix} B^{(m)}_{0,M_x}(x) \\ B^{(m)}_{1,M_x}(x) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ B^{(m)}_{M_x,M_x}(x) \end{bmatrix} = \frac{\prod_{k=1}^{m}(M_x-k+1)}{(x_{M_x}-x_0)^m} \begin{bmatrix} (-1)^m & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (-1)^{m-1}\binom{m}{1} & (-1)^m & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (-1)^{m-2}\binom{m}{2} & (-1)^{m-1}\binom{m}{1} & (-1)^m & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ (-1)^{m-3}\binom{m}{3} & (-1)^{m-2}\binom{m}{2} & (-1)^{m-1}\binom{m}{1} & (-1)^m & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ (-1)^{m-4}\binom{m}{4} & (-1)^{m-3}\binom{m}{3} & (-1)^{m-2}\binom{m}{2} & (-1)^{m-1}\binom{m}{1} & (-1)^m & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\binom{m}{1} & \binom{m}{2} & -\binom{m}{3} & \binom{m}{4} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\binom{m}{1} & \binom{m}{2} & -\binom{m}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\binom{m}{1} & \binom{m}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\binom{m}{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} B_{0,M_x-m}(x) \\ B_{1,M_x-m}(x) \\ B_{2,M_x-m}(x) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ B_{M_x-m,M_x-m}(x) \end{bmatrix}$$

or $$\mathrm{B}^{(m)}(x) = D_m B_{M_x-m}(x), \tag{2.8}$$

where $$\mathrm{B}^{(m)}(x) = \left[B^{(m)}_{0,M_x}(x), \quad B^{(m)}_{1,M_x}(x), \quad \cdots, \quad B^{(m)}_{M_x,M_x}(x) \right]^T,$$

$$B_{M_x-m}(x) = [\, B_{0,M_x-m}(x), \quad B_{1,M_x-m}(x), \quad \cdots, \quad B_{M_x-m,M_x-m}(x)\,]^T,$$

and D_m is the $(M_x+1)\times(M_x-m+1)$ operational matrix of derivative m.

2.2. The proposed numerical scheme

The approximate solution is defined by

$$u(x,t) = \sum_{j=0}^{M_t} \sum_{i=0}^{M_x} c_{j,i}\ B_{j,M_t}(t)\ B_{i,M_x}(x), \tag{2.9}$$

where $B_{j,M_t}(t)$ and $B_{i,M_x}(x)$ are the Bernstein polynomials and $c_{j,i}$ are the unknown coefficients.

Eq. (2.9) at point (x_i,t_j) with Kronecker product becomes

$$u(x_i,t_j) = (B(t_j)\otimes B(x_i))C, \tag{2.10}$$

where

$$B(t_j) = [\, B_{0,M_t}(t_j), \quad B_{1,M_t}(t_j), \quad \cdots, \quad B_{M_t,M_t}(t_j)\,], \tag{2.11}$$

$$B(x_i) = [\, B_{0,M_x}(x_i), \quad B_{1,M_x}(x_i), \quad \cdots, \quad B_{M_x,M_x}(x_i)\,], \tag{2.12}$$

and

$$C = [c_{0,0},\cdots,c_{0,M_x},c_{1,0},\cdots,c_{1,M_x},\cdots,c_{M_t,0},\cdots,c_{M_t,M_x}]^T. \tag{2.13}$$

The r^{th} order partial derivative of u with respect to x is defined by

$$\frac{\partial^r}{\partial x^r}u(x,t) = \sum_{j=0}^{M_t} \sum_{i=0}^{M_x} c_{j,i}\ B_{j,M_t}(t)\ B^{(r)}_{i,M_x}(x),$$

or

$$\frac{\partial^r}{\partial x^r}u(x_i,t_j) = (B(t_j)\otimes D_r B_{M_x-r}(x_i))C. \tag{2.14}$$

The s^{th} order partial derivative of u with respect to t is defined by

$$\frac{\partial^s}{\partial t^s}u(x,t) = \sum_{j=0}^{M_t} \sum_{i=0}^{M_x} c_{j,i}\ B^{(s)}_{j,M_t}(t)\ B_{i,M_x}(x),$$

or

$$\frac{\partial^s}{\partial t^s}u(x_i,t_j) = (D_s B_{M_t-s}(t_j)\otimes B(x_i))C. \tag{2.15}$$

The k^{th} and l^{th} order partial derivatives of u with respect to x and t are defined by

$$\frac{\partial^{k+l}}{\partial x^k \partial t^l}u(x,t) = \sum_{j=0}^{M_t} \sum_{i=0}^{M_x} c_{j,i}\ B^{(l)}_{j,M_t}(t)\ B^{(k)}_{i,M_x}(x),$$

or

$$\frac{\partial^{k+l}}{\partial x^k \partial t^l}u(x_i,t_j) = (D_l B_{M_t-l}(t_j)\otimes D_k B_{M_x-k}(x_i))C. \tag{2.16}$$

By the same way, we can compute any partial derivatives of the approximate solution u.

3. Implementation on GRLW

In this section, we apply our proposed scheme to solve the RLW and MRLW Eq. (1.1) which is

$$u_t + [1+q(q+1)u^q]u_x - \mu u_{xxt} = 0. \tag{3.1}$$

Using the above Section 2, From Eqs. (2.10), (2.14) – (2.16), Eq. (3.1) at the point (x_i,t_j) becomes

$$\begin{aligned}&(D_1 B_{M_t-1}(t_j)\otimes B(x_i))C \\ &\quad + \left[1+q(q+1)\left[(B(t_j)\otimes B(x_i))C\right]^q\right]\left[(B(t_j)\otimes D_1 B_{M_x-1}(x_i))C\right] \\ &\quad - \mu(D_1 B_{M_t-1}(t_j)\otimes D_2 B_{M_x-2}(x_i))C = 0,\end{aligned} \tag{3.2}$$

By computing Eq. (3.2) at all points (x_i, t_j) for $i = 1, 2, \cdots, M_x - 1$ and $j = 1, \cdots, M_t$, we get

$$\begin{aligned}
&\left([\mathcal{B}_{M_t-1}]_{2:M_t+1,:}\cdot D_1\otimes[\mathcal{B}_x]_{2:M_x,:}\right)\cdot C\\
&+\left[1+q(q+1)\left[\left([\mathcal{B}_t]_{2:M_t+1,:}\otimes[\mathcal{B}_x]_{2:M_x,:}\right)\cdot C\right]^q\right]^\circ\\
&\times\left(\left([\mathcal{B}_t]_{2:M_t+1,:}\otimes[\mathcal{B}_{M_x-1}]_{2:M_x,:}\cdot D_1\right)\cdot C\right)\\
&-\mu\left([\mathcal{B}_{M_t-1}]_{2:M_t+1,:}\cdot D_1\otimes[\mathcal{B}_{M_x-2}]_{2:M_x,:}\cdot D_2\right)\cdot C=\underset{-}{\mathbf{O}},
\end{aligned}\tag{3.3}$$

$[\mathcal{B}_{M_t-1}]_{2:M_t+1,:}$, $[\mathcal{B}_x]_{2:M_x,:}$, $[\mathcal{B}_t]_{2:M_t+1,:}$, ... and $[\mathcal{B}_{M_x-2}]_{2:M_x,:}$ in the above Eq. (3.3) are submatrices. $[\mathcal{B}_{M_t-1}]_{2:M_t+1,:}$ is the submatrix from row 2 to row $M_t + 1$ and all columns in matrix $\mathcal{B}_{M_t-1}$, similarly, the other submatrices can be represented. The symbol $\circ$ is the Hadamard product. $\underset{-}{\mathbf{O}}$ is $(M_t)(M_x - 1)$ zeros vector. The submatrices in Eq. (3.3) are defined by

$$\mathcal{B}_t=\begin{bmatrix} B_{0,M_t}(t_0) & B_{1,M_t}(t_0) & \cdots & B_{M_t,M_t}(t_0)\\ B_{0,M_t}(t_1) & B_{1,M_t}(t_1) & \cdots & B_{M_t,M_t}(t_1)\\ \vdots & \vdots & \vdots & \vdots\\ B_{0,M_t}(t_{M_t}) & B_{1,M_t}(t_{M_t}) & \cdots & B_{M_t,M_t}(t_{M_t})\end{bmatrix}^T,\tag{3.4}$$

$$\mathcal{B}_x=\begin{bmatrix} B_{0,M_x}(x_0) & B_{1,M_x}(x_0) & \cdots & B_{M_x,M_x}(x_0)\\ B_{0,M_x}(x_1) & B_{1,M_x}(x_1) & \cdots & B_{M_x,M_x}(x_1)\\ \vdots & \vdots & \vdots & \vdots\\ B_{0,M_x}(x_{M_x}) & B_{1,M_x}(x_{M_x}) & \cdots & B_{M_x,M_x}(x_{M_x})\end{bmatrix}^T,\tag{3.5}$$

$$\mathcal{B}_{M_t-1}=\begin{bmatrix} B_{0,M_t-1}(t_0) & B_{1,M_t-1}(t_0) & \cdots & B_{M_t-1,M_t-1}(t_0)\\ B_{0,M_t-1}(t_1) & B_{1,M_t-1}(t_1) & \cdots & B_{M_t-1,M_t-1}(t_1)\\ \vdots & \vdots & \vdots & \vdots\\ B_{0,M_t-1}(t_{M_t}) & B_{1,M_t-1}(t_{M_t}) & \cdots & B_{M_t-1,M_t-1}(t_{M_t})\end{bmatrix}^T,\tag{3.6}$$

$$\mathcal{B}_{M_x-1}=\begin{bmatrix} B_{0,M_x-1}(x_0) & B_{1,M_x-1}(x_0) & \cdots & B_{M_x-1,M_x-1}(x_0)\\ B_{0,M_x-1}(x_1) & B_{1,M_x-1}(x_1) & \cdots & B_{M_x-1,M_x-1}(x_1)\\ \vdots & \vdots & \vdots & \vdots\\ B_{0,M_x-1}(x_{M_x}) & B_{1,M_x-1}(x_{M_x}) & \cdots & B_{M_x-1,M_x-1}(x_{M_x})\end{bmatrix}^T,\tag{3.7}$$

$$\mathcal{B}_{M_x-2}=\begin{bmatrix} B_{0,M_x-2}(x_0) & B_{1,M_x-2}(x_0) & \cdots & B_{M_x-2,M_x-2}(x_0)\\ B_{0,M_x-2}(x_1) & B_{1,M_x-2}(x_1) & \cdots & B_{M_x-2,M_x-2}(x_1)\\ \vdots & \vdots & \vdots & \vdots\\ B_{0,M_x-2}(x_{M_x}) & B_{1,M_x-2}(x_{M_x}) & \cdots & B_{M_x-2,M_x-2}(x_{M_x})\end{bmatrix}^T.\tag{3.8}$$

The system of nonlinear equations (3.3) consists of $(M_t)(M_x - 1)$ equations with $(M_x + 1)(M_t + 1)$ unknowns (the vector C), we add $(M_x + 1)$ initial conditions and $(2M_t)$ boundary conditions to Eq. (3.3), then we get $(M_x + 1)(M_t + 1)$ system of nonlinear equations with $(M_x + 1)(M_t + 1)$ unknowns. This system is solved by Newton – Raphson method with 0.00002 initial values vector to find the vector C, then we obtain the approximate solution by Eq. (2.10).

4. Convergence study

To solve the system of nonlinear equations (3.3) with the initial and boundary conditions by Newton – Raphson method, first, all the equations take the following general form:

$$F(C)=\underset{-}{\mathbf{O}},\tag{4.1}$$

where $F(C) = \left[f_1(C), f_2(C), f_3(C), \cdots, f_{(M_x+1)(M_t+1)}(C)\right]^T$, the vector $\underset{-}{\mathbf{O}}$ is $(M_t + 1)(M_x + 1)$ zeros vector and C is the vector in Eq. (2.13).

The solution of the system of nonlinear Eqs. (4.1) by using Newton – Raphson method take the general iterative form:

$$C_{n+1}=C_n-J^{-1}(C_n)\ F(C_n),\tag{4.2}$$

for the number of iteration $n = 0,\ 1,\ 2,\ 3,\ \cdots$.

The Jacobin matrix is given by

$$J(C)=\begin{bmatrix}
\frac{\partial f_1(C)}{\partial c_{0,0}} & \cdots & \frac{\partial f_1(C)}{\partial c_{0,M_x}} & \frac{\partial f_1(C)}{\partial c_{1,0}} & \cdots & \frac{\partial f_1(C)}{\partial c_{1,M_x}} & \cdots & \frac{\partial f_1(C)}{\partial c_{M_t,M_x}}\\
\frac{\partial f_2(C)}{\partial c_{0,0}} & \cdots & \frac{\partial f_2(C)}{\partial c_{0,M_x}} & \frac{\partial f_2(C)}{\partial c_{1,0}} & \cdots & \frac{\partial f_2(C)}{\partial c_{1,M_x}} & \cdots & \frac{\partial f_2(C)}{\partial c_{M_t,M_x}}\\
\frac{\partial f_3(C)}{\partial c_{0,0}} & \cdots & \frac{\partial f_3(C)}{\partial c_{0,M_x}} & \frac{\partial f_3(C)}{\partial c_{1,0}} & \cdots & \frac{\partial f_3(C)}{\partial c_{1,M_x}} & \cdots & \frac{\partial f_3(C)}{\partial c_{M_t,M_x}}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\frac{\partial f_{(M_x+1)(M_t+1)}(C)}{\partial c_{0,0}} & \cdots & \frac{\partial f_{(M_x+1)(M_t+1)}(C)}{\partial c_{0,M_x}} & \frac{\partial f_{(M_x+1)(M_t+1)}(C)}{\partial c_{1,0}} & \cdots & \frac{\partial f_{(M_x+1)(M_t+1)}(C)}{\partial c_{1,M_x}} & \cdots & \frac{\partial f_{(M_x+1)(M_t+1)}(C)}{\partial c_{M_t,M_x}}
\end{bmatrix}\tag{4.3}$$

The inverse of Jacobin matrix (4.3) at any iteration exists if the determinant of $J(C_n)$ is nonzero, this means that $J(C_n)$ is nonsingular matrix (i. e. $det(J(C_n)) = |J(C_n)| \neq 0$).

Hence, for suitable choice for the initial values vector C_0 the iterative Eq. (4.2) converges.

5. Numerical calculations

The error norms $\|E\|_2$ and $\|E\|_\infty$ are given by

$$\|E\|_2=\left[\sum_{i=0}^{M_x}h_x\left|U_{i,j}-u_{i,j}\right|^2\right]^{\frac{1}{2}},\tag{5.1}$$

$$\|E\|_\infty=\max_{0\le i\le M_x}\left|U_{i,j}-u_{i,j}\right|,\tag{5.2}$$

Table 1
Numerical values of I_1, I_2, I_3, $\|E\|_2$ and $\|E\|_\infty$ for single solitary waves of RLW equation.

	I_1	I_2	I_3	$\|E\|_2$	$\|E\|_\infty$
Exact values	1.989975	0.202616	0.644750		
t					
0	1.905616	0.201597	0.641899	0	0
2	1.922090	0.202107	0.643325	0.000838	0.000661
4	1.937151	0.202273	0.643722	0.000921	0.000493
6	1.922198	0.202864	0.645895	0.003252	0.001680
8	1.964870	0.204673	0.651771	0.008991	0.005348
10	1.957609	0.204165	0.649984	0.006444	0.002655

Table 2
Numerical values of I_1, I_2, I_3, $\|E\|_2$ and $\|E\|_\infty$ for single solitary waves of MRLW equation.

	I_1	I_2	I_3	$\|E\|_2$	$\|E\|_\infty$
Exact values	3.294930	0.683426	0.024121		
t					
0	3.185491	0.682050	0.024235	0	0
2	3.172303	0.682068	0.023539	0.062203	0.027067
4	3.143492	0.680985	0.022880	0.121555	0.053836
6	3.223022	0.688285	0.023067	0.179312	0.079366
8	3.229971	0.688489	0.022181	0.229782	0.098547

Table 3
Comparisons of $\|E\|_2$ and $\|E\|_\infty$ for single solitary waves of RLW equation.

	Present method $M_x = M_t = 30$		CFDM [11] $M_x = 1001$ & $M_t = 101$		C-C SCM [4] $M_x = M_t = 34$	
t	$\|E\|_2$	$\|E\|_\infty$	$\|E\|_2$	$\|E\|_\infty$	$\|E\|_2$	$\|E\|_\infty$
2	0.000838	0.000661	0.013774	0.005403	0.000460	0.000531
4	0.000921	0.000493	0.012347	0.004610	0.000427	0.000485
6	0.003252	0.001680	0.010985	0.003841	0.000320	0.000348
8	0.008991	0.005348	0.009737	0.003158	0.000577	0.000855
10	0.006444	0.002655	0.008677	0.002656	0.000295	0.000269

Table 4
Comparisons of $\|E\|_2$ and $\|E\|_\infty$ for single solitary waves of MRLW equation.

	Present method $M_x = 28$ & $M_t = 10$		CFDM [11] $M_x = 1001$ & $M_t = 101$		C-C SCM [4] $M_x = M_t = 34$	
t	$\|E\|_2$	$\|E\|_\infty$	$\|E\|_2$	$\|E\|_\infty$	$\|E\|_2$	$\|E\|_\infty$
2	0.062203	0.027067	0.039859	0.018973	0.004992	0.007067
4	0.121555	0.053836	0.036136	0.015780	0.004157	0.005363
6	0.179312	0.079366	0.032839	0.013296	0.003009	0.003749
8	0.229782	0.098547	0.030230	0.011791	0.013190	0.027966

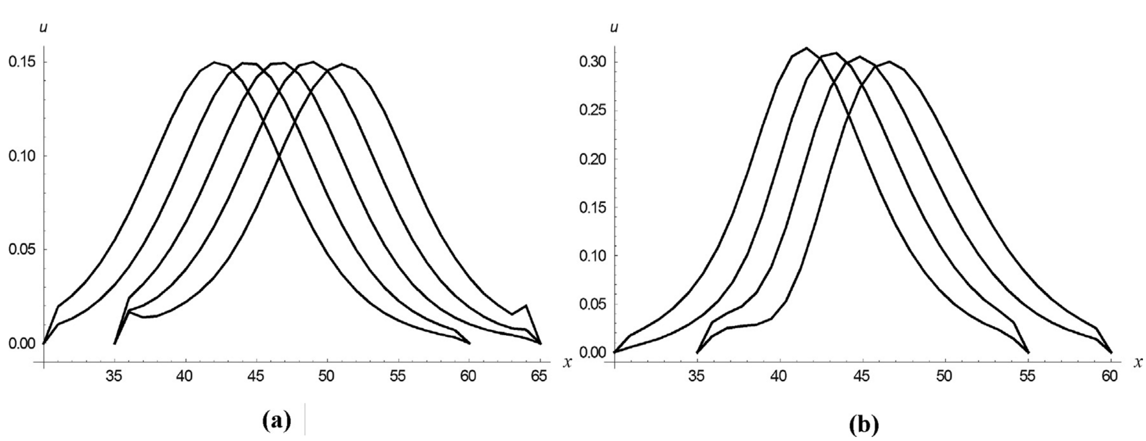

Fig. 1. The single solitary waves for **(a)** RLW at $t = 2,\ 4,\ 6,\ 8$ and 10, **(b)** MRLW at $t = 2,\ 4,\ 6$ and 8.

Table 5
Numerical values of I_1, I_2, I_3, $\|E\|_2$ and $\|E\|_\infty$ for the interaction of two solitary waves of RLW equation.

	I_1	I_2	I_3	$\|E\|_2$	$\|E\|_\infty$
Exact values	4.929360	0.810089	2.690570		
t					
0	4.876714	0.828941	2.760215	0	0
2	4.922790	0.833894	2.775963	0.096800	0.034412
4	4.781084	0.816752	2.711031	0.165736	0.065432
6	4.839147	0.826705	2.745497	0.251905	0.096595
8	4.773788	0.825988	2.738019	0.325612	0.124557
10	4.706124	0.824743	2.730428	0.393517	0.147443

Table 6
Numerical values of I_1, I_2, I_3, $\|E\|_2$ and $\|E\|_\infty$ for the interaction of two solitary waves of MRLW equation.

	I_1	I_2	I_3	$\|E\|_2$	$\|E\|_\infty$
Exact values	6.736370	1.717650	0.100327		
t					
0	6.709516	1.738899	0.103261	0	0
2	6.172924	1.797366	0.043418	0.316567	0.189630
4	6.805982	1.761075	0.085794	0.374344	0.146518
6	6.477652	1.670898	0.061699	0.468016	0.182732
8	6.379932	1.659413	0.049539	0.581737	0.214964
10	6.246313	1.653736	0.033816	0.677125	0.236274

where $U_{i,j}$ is the analytical solution in Eq. (1.2) at (x_i, t_j) and$u_{i,j} = u(x_i, t_j)$ is the numerical solution in Eq. (2.10).

The invariants of motion for RLW equation are given by [4,11]

$$\mathrm{I}_1 = \int_{x_0}^{x_{M_x}} u \, dx \simeq h_x \sum_{i=0}^{M_x} u_{i,j}, \tag{5.3}$$

$$\mathrm{I}_2 = \int_{x_0}^{x_{M_x}} [u^2 + \mu \, u_x^2] dx \simeq h_x \sum_{i=0}^{M_x} \left[u_{i,j}^2 + \mu(u_x)_{i,j}^2\right], \tag{5.4}$$

$$\mathrm{I}_3 = \int_{x_0}^{x_{M_x}} [2u^3 + 3u^2] dx \simeq h_x \sum_{i=0}^{M_x} \left[2u_{i,j}^3 + 3u_{i,j}^2\right]. \tag{5.5}$$

The invariants of motion for MRLW equation are given by [4,11]

$$\mathrm{I}_1 = \int_{x_0}^{x_{M_x}} u \, dx \simeq h_x \sum_{i=0}^{M_x} u_{i,j}, \tag{5.6}$$

$$\mathrm{I}_2 = \int_{x_0}^{x_{M_x}} [u^2 + \mu \, u_x^2] dx \simeq h_x \sum_{i=0}^{M_x} \left[u_{i,j}^2 + \mu \, (u_x)_{i,j}^2\right], \tag{5.7}$$

$$\mathrm{I}_3 = \int_{x_0}^{x_{M_x}} [u^4 - \mu \, u_x^2] dx \simeq h_x \sum_{i=0}^{M_x} \left[u_{i,j}^4 - \mu \, (u_x)_{i,j}^2\right]. \tag{5.8}$$

I_1, I_2 and I_3 in the above equations represent mass, momentum and energy, respectively.

5.1. Single solitary waves

To solve the RLW and MRLW Eq. (1.1) by the B-CM, the initial and boundary conditions at selected collocation point (x_i, t_j) become

$$u(x_i, 0) = (B(0) \otimes B(x_i))C = A\left[\mathrm{sech}^2[BW(x_i - x_0)]\right]^{\frac{1}{q}}, \tag{5.9}$$

$$u(x_0, t_j) = (B(t_j) \otimes B(x_0))C = 0, \tag{5.10}$$

$$u(x_{M_x}, t_j) = (B(t_j) \otimes B(x_{M_x}))C = 0, \tag{5.11}$$

$$u_x(x_0, t_j) = (B(t_j) \otimes D_1 B_{M_x-1}(x_0))C = 0, \tag{5.12}$$

$$u_x(x_{M_x}, t_j) = (B(t_j) \otimes D_1 B_{M_x-1}(x_{M_x}))C = 0. \tag{5.13}$$

The exact values for RLW equation are given by [4,11]

$$\mathrm{I}_1 = \frac{6\upsilon}{Q}, \quad \mathrm{I}_2 = \frac{6\upsilon^2}{Q} + \frac{6\upsilon^2 \ \mu \ Q}{5} \quad \text{and} \quad \mathrm{I}_3 = \frac{18\upsilon^2}{Q} + \frac{72\upsilon^3}{5Q}. \tag{5.14}$$

The exact values for MRLW equation are given by [4,11]

$$\mathrm{I}_1 = \frac{\pi\sqrt{\upsilon}}{Q}, \quad \mathrm{I}_2 = \frac{2\upsilon}{Q} + \frac{2\upsilon\mu Q}{3} \quad \text{and} \quad \mathrm{I}_3 = \frac{4\upsilon^2}{3Q} - \frac{2\upsilon\mu Q}{3}. \tag{5.15}$$

Table 1 and Table 2 illustrate the numerical values of I_1, I_2, I_3, $\|E\|_2$and $\|E\|_\infty$ for RLW and MRLW equation, respectively, at $x_0 = 40$, $\mu = 1$, $\upsilon = 0.1$and $x \in [30,65]$. We take $M_x = M_t = 30$ for RLW equation (the 3rd iteration of Newton – Raphson method is taken), but we take $M_x = 28$ and $M_t = 10$ for MRLW equation (the 1st iteration of Newton – Raphson method is taken). Tables 3 and 4 consiste of comparisons of $\|E\|_2$ and $\|E\|_\infty$ for single solitary waves of RLW and MRLW equation between B-CM and other methods, respectively. In Table 3, the error norms of single solitary waves of RLW for B-CM are less than the error norms for CFDM [11] and the error norms for B-CM are consistent with the error norms for C-C SCM [4], but in Table 4, the error norms of single solitary waves of MRLW for B-CM are consistent with the error norms for CFDM [11] and the error norms for B-CM are greater than the error norms for C-C SCM [4]. Fig. 1 presents the motion of single solitary waves at $t = 2, 4, 6, 8$ and 10 for RLW equation (Fig. 1 (a)) and at $t = 2, 4, 6$ and 8 for MRLW equation (Fig. 1 (b)). From Fig. 1, we observed that the speed remains fixed when the soliton moves to the right through the space range $x \in [0, 100]$

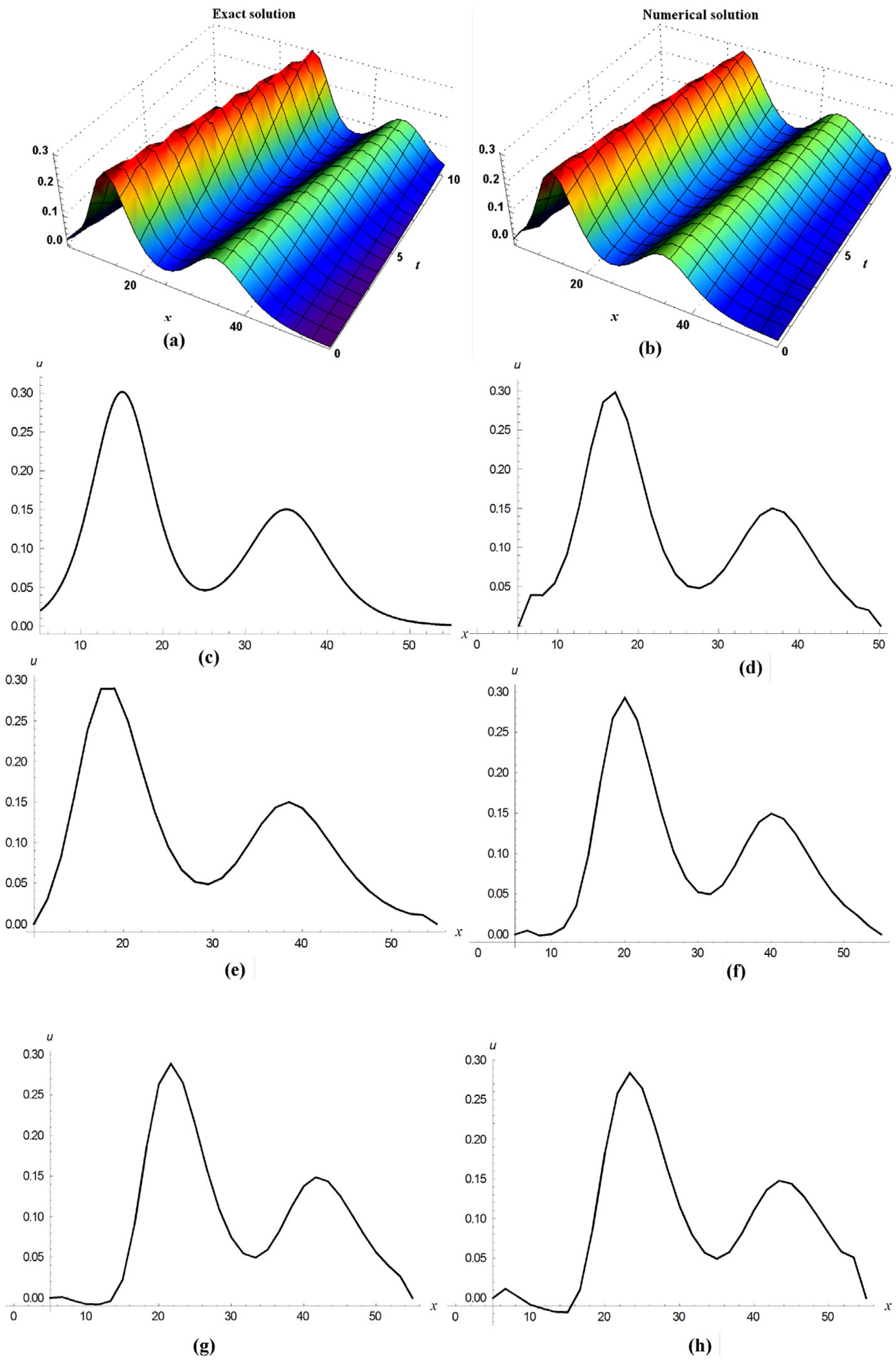

Fig. 2. Interaction of two solitary waves for RLW equation at **(g)**: (**a**) Exact solution, (**b**) Numerical solution (**c**) $t = 0$, (**d**) $t = 2$, (***e***) $t = 4$ and (**f**) $t = 6$, (**g**) $t = 8$ and **(h)** $t = 10$.

and the amplitude is nearly unchaged as the time t increases, accordingly, the amplitude of the single solitary waves for RLW and MRLW equation are 0.15 and 0.316228, respectively.

5.2. Interaction of two solitary waves

The initial condition of the RLW and MRLW Eq. (1.1) at selected collocation point (x_i, t_j) are given by

$$u(x_i, 0) = (B(0) \otimes B(x_i))C = \sum_{l=1}^{2} A_l \left[\text{sech}^2[BW_l(x_i - x_l)]\right]^{\frac{1}{q}}, \tag{5.16}$$

where $A_l = \left[\frac{\upsilon_l(q+2)}{2q}\right]^{\frac{1}{q}}$, $BW_l = \frac{q}{2} Q_l$, $Q_l = \sqrt{\frac{\upsilon_l}{\mu(\upsilon_l+1)}}$ for $l = 1,\ 2.$ υ_l and x_l are positive numbers.

The exact values for the interaction of two solitary waves for RLW equation are computed by [4,11]

$$I_1 = \sum_{l=1}^{2} \frac{6\upsilon_l}{Q_l}, \quad I_2 = \sum_{l=1}^{2} \left(\frac{6\upsilon_l^2}{Q_l} + \frac{6\upsilon_l^2\ \mu\ Q_l}{5}\right) \quad \text{and} \quad I_3 = \sum_{l=1}^{2} \left(\frac{18\upsilon_l^2}{Q_l} + \frac{72\upsilon_l^3}{5Q_l}\right). \tag{5.17}$$

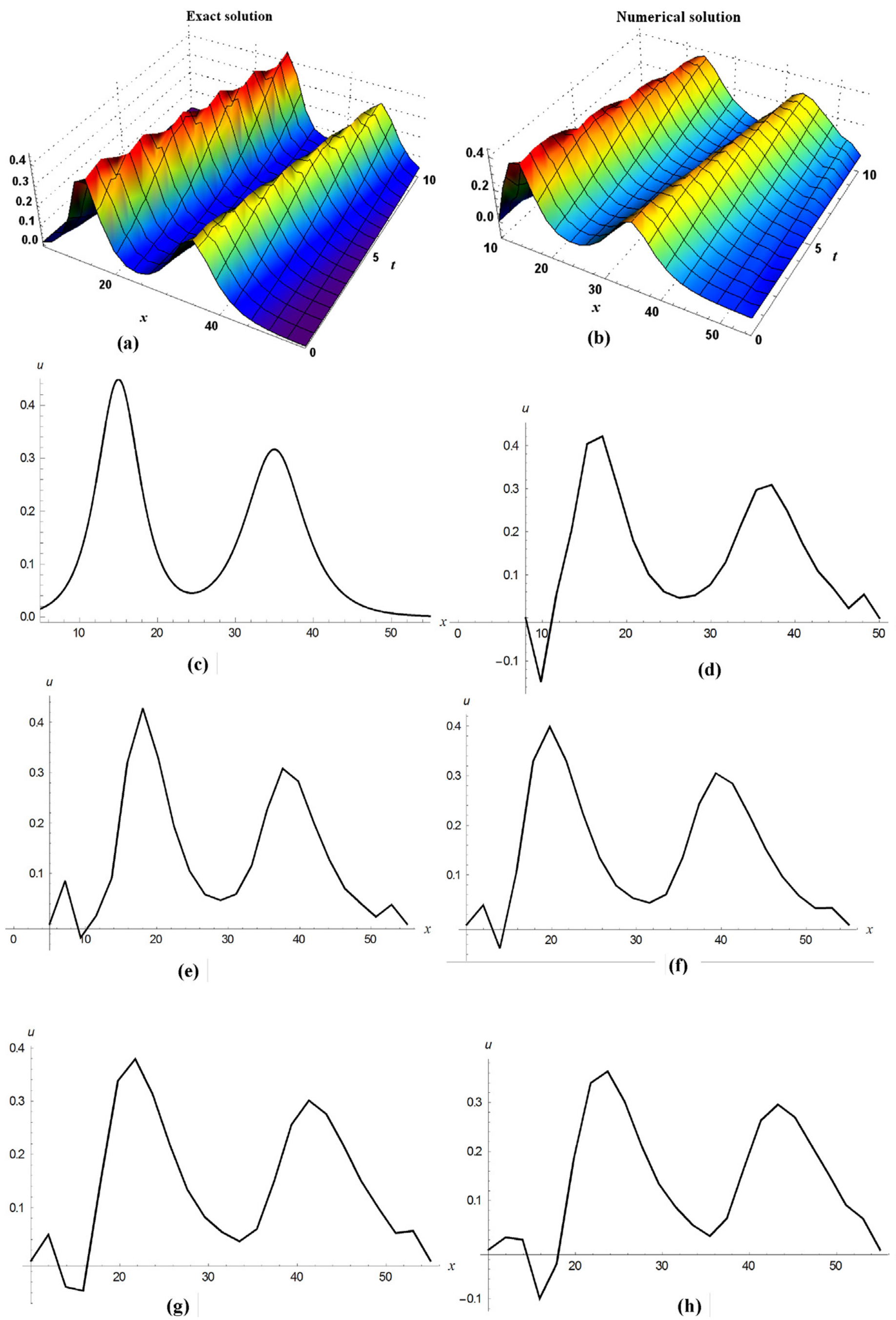

Fig. 3. Interaction of two solitary waves for MRLW equation: **(a)** Exact solution, **(b)** Numerical solution **(c)** $t = 0$, **(d)**$t = 2$, **(e)** $t = 4$ and **(f)** $t = 6$. **Part II** Interaction of two solitary waves for MRLW equation at **(g)** $t = 8$ and **(h)** $t = 10$.

The exact values for the interaction of two solitary waves for MRLW equation are calculated by [4,11]

$$I_1 = \sum_{l=1}^{2} \frac{\pi\sqrt{\upsilon_l}}{Q_l}, \quad I_2 = \sum_{l=1}^{2} \left(\frac{2\upsilon_l}{Q_l} + \frac{2\upsilon_l \mu Q_l}{3}\right) \quad \text{and} \quad I_3 = \sum_{l=1}^{2} \left(\frac{4\upsilon_l^2}{3Q_l} - \frac{2\upsilon_l \mu Q_l}{3}\right). \tag{5.18}$$

Table 5 and Table 6 summarize the numerical values of $I_1, I_2, I_3, \|E\|_2$and $\|E\|_\infty$ for RLW and MRLW equation, respectively, at $x_1 = 15,\ x_2 = 35,\ \upsilon_1 = 0.2,\ \upsilon_2 = 0.1,\ \mu = 1$ and $x \in [0, 55]$. We take $M_x = 30$ and $M_t = 10$ for RLW equation but we take $M_x = 23$ and $M_t = 11$ for MRLW equation (the 1st iteration of Newton – Raphson method is taken for both RLW and MRLW equation). Fig. 2 and Fig. 3 illustrate the interaction of two solitary waves for RLW and MRLW equation at $t = 0,\ 2,\ 4,\ 6,\ 8$ and 10, respectively.

6. Conclusion

The Bernstein polynomials in both the space and time directions have been applied to solve the RLW equation and the MRLW equation. We used the Kronecker product and Hadamard product, hence the equations became in a matrix form so they are more easy and simple to use. The proposed scheme (B-CM) leads to reduce the GRLW equations to system of nonlinear algebraic equations which has been solved numerically by Newton – Raphson method. Convergence study for the proposed scheme is also presented. The above numerical outcomes and comparisons for the RLW and MRLW equations show that the B-CM is qualified to solve RLW and MRLW equation. At the end, the B-CM is qualified to solve NPDEs, integral and integro-differential equations.

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